Tentamen Analyse 2012

Datum : 06-11-2012
Tijd : 14.00 - 17.00, Tentamenhal 1

You need to clearly provide arguments for all your answers; 'yes' or 'no' answers are not allowed.
The detailed grading scheme can be found below.

1. (a) Consider the sequence (a_n) defined by

$$a_n = (-1)^n \frac{2n+1}{n+2}, \quad n = 1, 2, \ldots$$

   Is (a_n) convergent ? Determine the limit points of the set $A := \{a_1, a_2, \ldots\}$, and determine the closure $\overline{A}$.

   (b) Take any non-empty subset $E \subset \mathbb{R}$. Define $G := E^c$ (the complement of E), and $F := \overline{G}$. Show that F^c is the largest open set contained in E.
   Does there exist for any E a *smallest* open set *containing* E ? (If yes, prove; if not, give a counterexample.)

2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function with

$$f(a) \leq a, \quad f(b) \geq b$$

   Show that there exists a point $c \in [a, b]$ with $f(c) = c$.

3. Consider a differentiable function $g : \mathbb{R} \to \mathbb{R}$ with bounded derivative g' (that is, there exists an $M > 0$ such that $|g'(x)| \leq M$ for all $x \in \mathbb{R}$). Prove that the function $f_\epsilon : \mathbb{R} \to \mathbb{R}$ defined as

$$f_\epsilon(x) := x + \epsilon g(x)$$

   is one-to-one for small enough $\epsilon > 0$.

4. Consider the sequence of functions

$$f_n(x) = \frac{x^n}{1 + x^n}, \quad x \geq 0$$

   (a) Verify by computation that for each x the sequence $f_n(x)$ converges to $f(x)$, where the function $f : [0, \infty) \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 0, & x \in [0, 1) \\ \frac{1}{2}, & x = 1 \\ 1, & x \in (1, \infty) \end{cases}$$

   Deduce from this that f_n does not converge uniformly on $[0, 1]$.

(b) Show that f_n converges uniformly on every interval $[0, c]$ with $c < 1$.

(c) Show that f_n converges uniformly on every interval $[b, \infty)$ with $b > 1$. Does f_n converge uniformly on the interval $(1, \infty)$? If yes, prove; if not, show why. (**Hint**: You may use $(1 + \frac{1}{n})^n \to e$ for $n \to \infty$.)

5. Consider the functions $f_n : [0, 1] \to \mathbb{R}$ defined by

$$f_n(x) = n^2 x^n (1 - x), \quad x \in [0, 1]$$

   (a) Verify that f_n converges pointwise to the zero-function $f(x) = 0, x \in [0, 1]$.
   (b) Calculate $\lim_{n\to\infty} \int_0^1 f_n(x)dx$.
   (c) Does $f_n \to f$ uniformly on $[0, 1]$? (Provide arguments.)

6. Consider the series

$$f(x) = \sum_{n=2}^{\infty} \frac{1}{\sin x + n^2}$$

   (a) Prove that f is continuous on $\mathbb{R}$.
   (b) Is f differentiable on $\mathbb{R}$? If so, is f' continuous ? (Again, provide argumentation.)

7. Consider a continuous function $f : [a, b] \to \mathbb{R}$. Prove that there exists a $c \in (a, b)$ such that

$$f(c) = \frac{1}{b - a} \int_a^b f(x)dx$$

   Does this still hold if $f : [a, b] \to \mathbb{R}$ is bounded, but continuous only on (a, b) ?

Grading scheme (Total 100). Free 10.

1. a: 6, b; 7.
2. 10.
3. 12.
4. a: 6, b: 4, c: 9.
5. a: 2, b: 5, c: 4.
6. a: 5, b: 9.
7. 11.